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Linear Functions Revisited

(Joint work with Benjamin Doerr and Daniel Johannsen)

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March 24, 2010

Contents

- **"OneMax is fastest"**
- Universal drift functions

Main Results

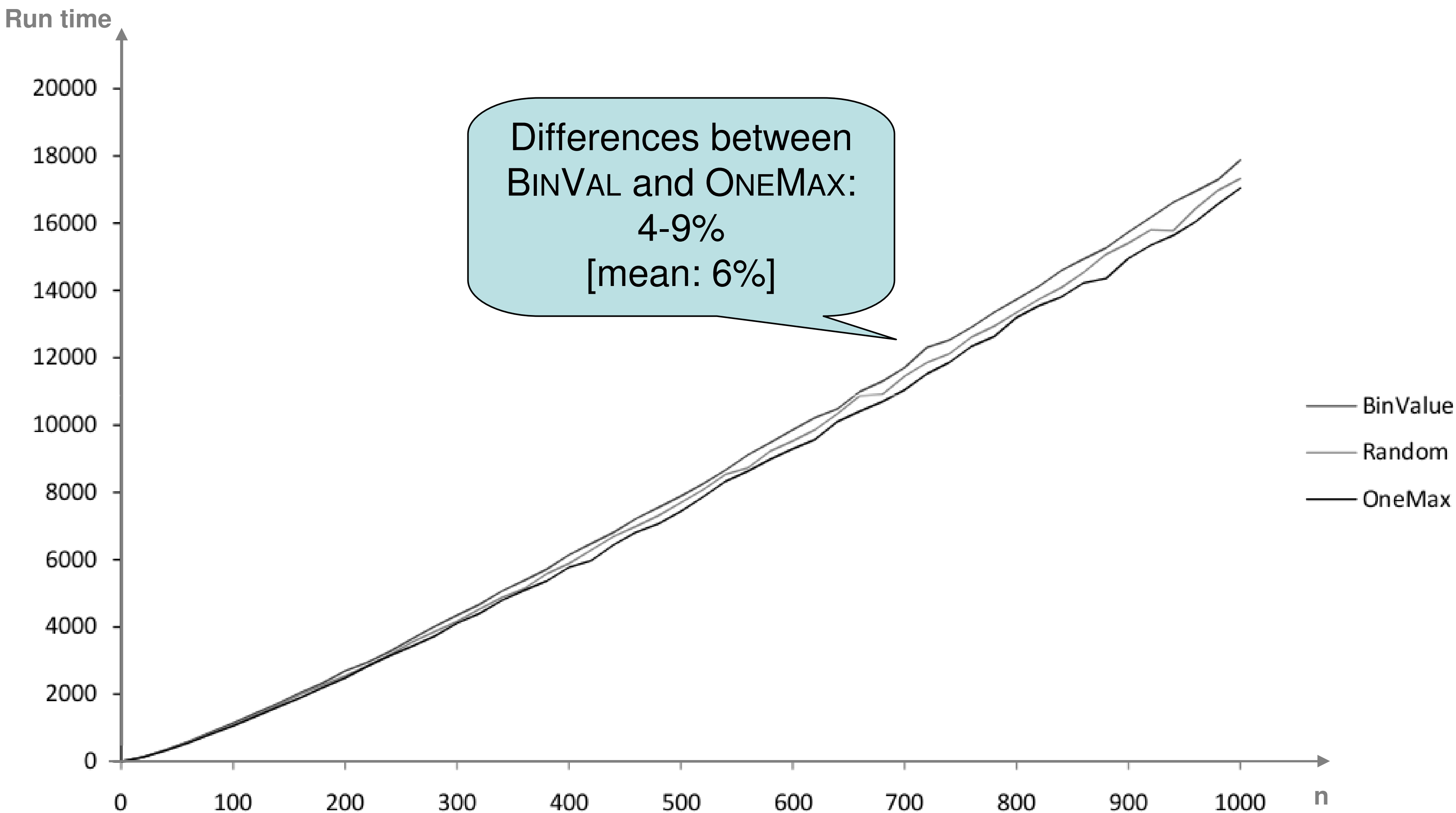
"ONEMAX is fastest" [Doerr, Johannsen, W]	Let f be a function such that 0^n is the only optimal solution, i.e., $\{x \mid f(x) \text{ minimal}\} = \{0^n\}$ Then $E[T^f] \geq E[T^{\text{ONEMAX}}]$

Lower bound [Doerr, Fouz, Witt]	For all $n \in \mathbb{N}$ the (1+1) EA needs at least $(1-o(1))\ e\ n \ln(n)$ steps to optimize ONEMAX.
Upper bound [Doerr, Johannsen, W]	The (1+1) EA optimizes any linear f in $(1+o(1))\ 1.39\ e\ n \ln(n)$ steps.

→ Running time bounds reduced to a 39% gap

Experimental Results

Average optimization time of the (1+1) EA for different linear functions[1]



1: After 1,000 iterations of each problem instance

Proof Sketch

"ONEMAX is fastest" [Doerr, Johannsen, W]	Let f be a function such that 0^n is the only optimal solution, i.e., $\{x \mid f(x) \text{ minimal}\} = \{0^n\}$ Then $E[\, T^f \,] \geq E[\, T^{\text{ONEMAX}} \,]$

- Let $T^f_k := \min\{\, T^f(x) \mid x \in \{0,1\}^n \text{ with } |x|_1 \geq k \,\}$
- Let $T`_k := \min\{\, T^{\text{ONEMAX}}(x) \mid x \in \{0,1\}^n \text{ with } |x|_1 = k \,\}$
- Use the following facts
  - $\Pr[\, |mut(x)|_1 = i \,] \leq \Pr[\, |mut(y)|_1 = i \,]$ whenever $|y|_1 \geq |x|_1$
  - $\Pr_{\text{ONEMAX}}[\, |x^+|_1 = i \,] \geq \Pr_f[\, |x^+|_1 = i \,]$ for all $i \leq |x|_1$
  - induction hypothesis

  to derive

$$T^f_{k+1} \geq 1 + \sum_{i \leq k} \Pr[\, |x^+|_1 = i \,]\, T`_i + \Pr[\, |x^+|_1 > k \,]\, T^f_{k+1} \qquad (1)$$

- Observe that $T`_{k+1}$ fulfills (1) with equality
- Now, $T^f(x) \geq T^f_{|x|} \geq T`_{|x|} = T^{\text{ONEMAX}}(x)$ for all x. Hence, $E[\, T^f \,] \geq E[\, T^{\text{ONEMAX}} \,]$

Outlook – Runtime of the (1+1) EA on linear functions

- Close the gap between
  $(1-o(1))\, e\, n \ln(n)$ and $(1+o(1))\, 1.39\, e\, n \ln(n)$
- Prove strict superiority of the runtime for OneMax
- Similar results for mutation probabilities other than 1/n

Contents

- "OneMax is fastest"
- **Universal drift functions**

Universal Linear Drift Functions

- Reminder [alternative proof in our paper]:

Theorem [Existence]	Let $g: \{0,1\}^n \longrightarrow \mathbb{R}, x \mapsto \sum_{i \le n/2} x_i + 5/4 \sum_{i > n/2} x_i$.
	Then, g serves as a drift function for all linear f with monotone weights[1].

- A few things to note:
  - Proof mainly relies on the fact that for p=1/n, the probability that more than 1 bits flip is exponentially decreasing
  - So, what happens if p > 1/n? [remainder of the talk]

Definition [Universal]	We call g a *universal linear drift function* if (i) g is a drift function for <u>all</u> linear f with monotone weights (ii) g is linear itself

1: I.e. $f: \{0,1\}^n \longrightarrow \mathbb{R}, x \mapsto \sum f_i x_i$ with $f_1 \le \ldots \le f_n$

Nonexistence of Universal Linear Drift Functions for p>1/n

Setting	Nonexistence of universal linear drift functions for
Multiplicative Drift Theorem	$p \geq 1.95/n$
Classical Drift Theorem	$p \geq 3.3/n$ [further ideas to improve this bound exist]
Jägersküpper	$p \geq 7/n$ [further ideas to improve this bound exist]

On this slide, p denotes the mutation probability [i.e., the probability that the (1+1) EA_p flips a given bit in the mutation step].

Proof Sketch

Theorem [Nonexistence in multiplicative setting]	Let $p \geq 1.95/n$. For all linear $g: \{0,1\}^n \longrightarrow \mathbb{R}$, $x \mapsto \sum g_i x_i$ with $1 = g_1 \leq \ldots \leq g_n$ there exist • a point $x \in \{0,1\}^n$ and • a linear function $f: \{0,1\}^n \longrightarrow \mathbb{R}$, $y \mapsto \sum f_i y_i$ with $f_1 \leq \ldots \leq f_n$ such that the expected drift of g in x with respect to f is ≤ 0.

- Assume the contrary
  (i.e., assume that there exists a linear g such that $E[D(g,f,x)]>0$ for all f, x)
- Show
  A. Upper bound $\Sigma g_i \leq (n-1)+1/p$
     [evaluate expected drift for f=OneMax and $x=e_1=(0,\ldots,0,1)$]
  B. Lower bounds $g_i \geq p \, \Sigma_{k<i} \, g_k$
     [evaluate expected drift for f=BinVal and $x=e_i=(0,\ldots,0,1,0,\ldots,0)$]
- Derive contradiction
  (algebraically for $p \geq 2.2$, using MATLAB for $p \geq 1.95$)

Proof for other settings follow by applying the same ideas

Outlook – Universal Linear Drift Functions

- For the different settings: Determine c such that universal linear drift functions exists if and only if $p<c/n$
  [by now, we have
  $c < 1.95$ in the multiplicative setting,
  $c \leq 3.3$ in the classical setting, and
  $c < 7$ in the "Jägersküpper" setting]

- Existence of drift theorems with universal drift functions?
  [probably using more information about the distribution of the search points]